COROLLARY. *If in (19) the x's are rational integers and $\mathfrak{p} = (p)$ with p a prime rational integer then this congruence always has k solutions for p sufficiently large with the other conditions in Theorem II holding, k any integer.*

This corollary was given by Mordell[1] when $\alpha_{s+1} \not\equiv 0$, and we include *all*[3] solutions of (19). We considered only solutions prime to $\mathfrak{p}$ (primitive solutions) in our work, following the conditions (2) and (20).

[1] These PROCEEDINGS, 33, 236–242 (1947). In this paper reference to previous results was made, but an important paper by Mordell, *Mathematische Zeitschrift*, 37, 207 (1933) which bore more directly on the contents, was, unfortunately, not mentioned. Other relevant references are Pellet, *Bull. Math. Soc. France*, 15, 80–93 (1886); Dickson, *Crelle*, 135, 181–188 (1909); Hurwitz, *Crelle*, 136, 272–292 (1909); Mitchell, *Ann. Math., II*, 18, 120 (1917); Davenport, *Jour. London Math. Soc.*, 6, 49–54 (1931); Schur, I., *Jahresber. Deutsch. Math. Verein.*, 25, 114 (1916).

[1a] Cf. a paper by Hua, Loo-Keng, "On a Double Exponential Sum," soon to appear in the *Science Reports of Tsing Hua University*. An abstract is given in the *Science Record of the Acad. Sinica*, 1, Nos. 1–2.

[2] These PROCEEDINGS, 32, 47–52 (1946).

[3] Mordell's method of proof is different from either of those used in the present paper. In some ways there is quite a distinction between finding the primitive solutions of an equation in a finite field and finding all solutions. The congruence

$$x_1^2 + x_2^2 + \ldots + x_s^2 \equiv 0 \pmod 2$$

has no solutions in integers prime to 2, if s is odd, but evidently has solutions for some of the x's even. Again the congruence $x^7 + y^7 + 1 \equiv 0 \pmod{491}$ has no solutions in integers x and y prime to 491. But the congruence $x^7 + 1 \equiv 0 \pmod{491}$ obviously has solutions.

GROUPS, CATEGORIES AND DUALITY

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It has long been recognized that the theorems of group theory display a certain duality. The concept of a lattice gives a partial expression for this duality, in that some of the theorems about groups which can be formulated in terms of the lattice of subgroups of a group display the customary lattice duality between meet (intersection) and join (union). The duality is not always present, in the sense that the lattice dual of a true theorem on groups need not be true; for example, a Jordan Holder theorem holds for certain ascending well-ordered infinite composition series, but not for the corresponding descending series.[1] Moreover, there are other striking group theoretic situations where a duality is present, but is not readily expressible in lattice-theoretic terms.

As an example, consider the direct product $D = G \times H$ of two groups

G and H, together with its canonical homomorphisms $\gamma(g, h) = g$, $\eta(g, h) = h$ into the given factors G and H. The system $[\gamma:D \rightarrow G;\ \eta:D \rightarrow H]$ consisting of the direct product together with these homomorphisms is characterized, up to isomorphism, by the following property: given any other such system $[\gamma':D' \rightarrow G,\ \eta':D' \rightarrow H]$ for the same groups G and H, there is one and only one homomorphism $\pi:D' \rightarrow D$ such that $\gamma' = \gamma\pi$, $\eta' = \eta\pi$. Dually, the free product P of groups G and H is the "most general" group generated by subgroups isomorphic to G and H, respectively. This means that there are canonical homomorphisms $\alpha:G \rightarrow P$ and $\beta:H \rightarrow P$ of the factors into the corresponding subgroups of P. This system (P, α, β) is characterized by the following property: given any system $[\alpha':G \rightarrow P',\ \beta':H \rightarrow P']$ there is one and only one homomorphism $\sigma:P \rightarrow P'$ such that $\sigma\alpha = \alpha'$, $\sigma\beta = \beta'$. The theorem that the direct product of any two groups exists is thus dual to the theorem asserting the existence of the free product. The *proofs* of these two theorems are not dual, but the proofs of many other formal properties are dual, as for instance in the case of the associative law $(G \times H) \times K \cong G \times (H \times K)$. For the direct product D, the canonical homomorphisms γ and η are *homomorphisms onto* their respective ranges G and H; in the case of the free product P the canonical homomorphisms α and β are *isomorphisms into* P. The "dual" of a theorem about groups and homomorphisms is to be obtained by inverting the direction of each homomorphism, inverting the order of all products of homomorphisms and replacing homomorphisms onto by isomorphisms into.

For abelian groups the duality is more marked. A free abelian group F can be characterized in terms of homomorphisms of abelian groups by the following property:[2] for any homomorphism $\alpha:F \rightarrow A$ and any second homomorphism $\beta:B \rightarrow A$ onto the image group A there exists a homomorphism $\gamma:F \rightarrow B$ with $\beta\gamma = \alpha$. (The corresponding characterization applies also to free non-abelian groups.) An infinitely divisible abelian group D is one in which there exists for each $d \in D$ and each integer m a solution x of the equation $mx = d$. Any homomorphism of an abelian group A into D can be extended to any abelian group B containing A. This property characterizes the infinitely divisible abelian groups; it may be stated in a form dual to the characteristic property of free groups: given $\alpha:A \rightarrow D$ and an isomorphism $\beta:A \rightarrow B$ of A into B, there exists a $\gamma:B \rightarrow D$ with $\gamma\beta = \alpha$. For an abelian group, free products reduce to direct products. If a factor group of an abelian group is a free group, it is a direct factor. Dually, if a subgroup of an abelian group is infinitely divisible, it is a direct factor.

This duality for abelian groups appears in algebraic topology as a duality between homology and cohomology groups. This phenomenon is especially striking in the axiomatic form of homology theory.[3]

For locally compact topological abelian groups, the duality phenomena can be formulated explicitly by means of character groups;[4] each theorem then gives a dual theorem about the character groups of those groups involved in the original theorem. It is instructive to compare this formulation with the duality of plane projective geometry.[5] A pole-polar reciprocation gives a dual to each projective figure, comparable to the character group of a group. Alternatively, projective geometry has an "axiomatic" or "syntactical" duality: any theorem deducible from the incidence axioms remains true on the interchange of the primitive terms "point" and "line" in the statement of the theorem.

Our objective is a similar formulation of a (partial) axiomatic duality for groups. It clearly must concern the system consisting of all groups and all homomorphisms of one group into another. For certain other investigations of this and similar systems, Eilenberg and the author have introduced the notion of a category.[6] A *category* is a class of "mappings" (say, homomorphisms) in which the product $\alpha\beta$ of certain pairs of mappings α and β is defined. A mapping e is called an *identity* if $\rho\alpha = \alpha$ and $\beta\rho = \beta$ whenever the products in question are defined. These products must satisfy the axioms:

(C-1). *If the products $\gamma\beta$ and $(\gamma\beta)\alpha$ are defined, so is $\beta\alpha$;*

(C-1′). *If the products $\beta\alpha$ and $\gamma(\beta\alpha)$ are defined, so is $\gamma\beta$;*

(C-2). *If the products $\gamma\beta$ and $\beta\alpha$ are defined, so are the products $(\gamma\beta)\alpha$ and $\gamma(\beta\alpha)$, and these products are equal.*

(C-3). *For each γ there is an identity e_D such that γe_D is defined;*

(C-4). *For each γ there is an identity e_R such that $e_R\gamma$ is defined.*

It follows that the identities e_D and e_R are unique; they may be called, respectively, the *domain* and the *range* of the given mapping γ. A mapping θ with a two-sided inverse is an *equivalence.*

These axioms are clearly self dual, and a dual theory of free and direct products may be constructed in any category in which such products exist. These axioms do not, however, suffice to express the duality between "homomorphism onto" and "isomorphism into." These notions can be formulated in terms of subgroups and factor groups; with any subgroup $S \subset G$ we can associate the identity injection $i \cdot S \rightarrow G$ of S into G, and with any normal subgroup N of G we can associate the projection $\tau: G \rightarrow G/N$ mapping each element g of G into its coset gN in the factor group G/N. We propose to axiomatize the dual notions "injection" and "projection."

A *bicategory* is a category with two distinguished classes of mappings, the "injections" and the "projections," subject to the following self dual axioms:

(BC-1). *Every identity is both an injection and a projection;*

(BC-2). *The product of two injections (projections), when defined, is an injection (projection).*

(BC-3). *Every mapping γ can be represented uniquely as a product $\gamma = \kappa\theta\pi$, where π is a projection, θ an equivalence and κ an injection.*

A mapping of the form $\kappa\theta$ is called a mapping within (isomorphism into); one of the form $\theta\pi$ is called a mapping upon (homomorphism onto).

(BC-4). *The product of two mappings within (upon), when defined, is a mapping within (upon).*

(BC-5). *Two injections (projections) with identical domains and identical ranges are identical.*

These concepts suffice to give dual definitions of "subgroups" and "factor groups." Thus e_1 is a "subidentity" of ρ_2 if there exists an injection with domain e_1 and range e_2; this inclusion relation gives a partial order of the identities of a bicategory. We may then define a *lattice-ordered bicategory* as any bicategory in which the subidentities and factor identities of any given identity form a lattice under this partial order.

A group can be interpreted as a lattice-ordered bicategory with an identity; the mappings of the category are all equivalences, and are the elements of the group. A lattice L can be interpreted as a lattice-ordered bicategory in which all mappings are injections: the mappings of the category are the pairs $[a, b]$ with $a \supset b$, and with product $[a, b][b, c] = [a, c]$. Thus the concept "lattice-ordered bicategory" is a common generalization of the notions "group" and "lattice."

We contend that most of the phenomena of universal algebra and of (axiomatic) group duality[7] have appropriate and simple formulations in terms of lattice-ordered bicategories. In particular, for groups, one may use the lattice-ordered bicategory of all homomorphisms of one group into another. In this category we might interpret projection mapping to mean any (canonical) homomorphism $\tau: G \rightarrow G/N$ of a group G upon its factor group G/N. For this interpretation the product of two projections is not a projection (axiom BC-2 fails). This axiom might be saved by calling a projection any product of such canonical homomorphisms τ, but in this case the projection factor π of any homomorphism is not unique (axiom BC-3 fails).

This apparent difficulty can be surmounted by an attention to fundamentals. A factor group G/N may be described either as a group in which the *elements* are cosets of N, and the *equality* of elements is the equality of sets, or as a group in which the *elements* are the elements of G, and the "equality" is congruence modulo N. Both approaches are rigorous[8] and can be applied systematically (and with approximately equal inconvenience!) throughout group theory. The difficulties cited disappear when we adopt the second point of view, and regard a group G as a system of elements G with a reflexive symmetric and transitive "equality" relation such that logically identical elements are equal (but not necessarily conversely) and such that products of equal elements are equal.

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[1] Birkhoff, G., "Lattice Theory," *Am. Math. Soc. Colloq. Pub.*, **25**, 48 (1940).

[2] For the case of abelian groups with operators from a group Q, this property is used in Eilenberg, S., and MacLane, S., "Homology Theory of Spaces with Operators II," forthcoming in *Trans. Am. Math. Soc.*

[3] Eilenberg, S., and Steenrod, N., PROC. NAT. ACAD. SCI., **31**, 117–120 (1945). (The writer has also profited by reading further unpublished work of these authors on this subject.)

[4] Pontrjagin, L., *Topological Groups*, Princeton, 1939. Weil, A., *L'Integration dans les groupes topologiques et ses applications*, Paris, 1938.

[5] Veblen, O., and Young, J. W., *Projective Geometry*, Boston, 1910.

[6] Eilenberg, S., and MacLane, S., PROC. NAT. ACAD. SCI., **28**, 537–543 (1942); *Trans. Am. Math. Soc.*, **58**, 231–294 (1945).

[7] The formulation with bicategories does not yet indicate the duality between center and factor commutator groups, and similar dual concepts of verbal and marginal subgroups; Hall, P., *J. f. d. reine und angew. Math.*, **182**, 156–157 (1940).

[8] A careful treatment, emphasizing the equality approach, appears in the unjustly neglected book by Haupt, O., *Einführung in die Algebra*, Leipzig, 1929.

METHODS OF SYMMETRY AND CRITICAL POINTS OF HARMONIC FUNCTIONS

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The most powerful method known for the study of the location of the critical points of harmonic functions is the expression of the gradient of a given harmonic function as the force in a field due to a suitable distribution of matter.[1] Nevertheless simpler methods involving less machinery, based on topological considerations involving symmetry, yield some surprisingly deep results, as we wish to indicate in the present note. Our principal result is

THEOREM 1. *Denote by Π_1 and Π_2 the open upper and lower half-planes respectively. Let $u(x, y)$ be harmonic in a region R cut by the axis of reals, and let the relation*

$$u(x, y) > u(x, -y) \text{ for } (x, y) \text{ in } \Pi_1 \tag{1}$$

hold whenever both (x, y) and $(x, -y)$ lie in R. Then $u(x, y)$ has no critical point in R on the axis of reals.

Alternate sufficient conditions that $u(x, y)$ have no critical point in R on the axis of reals are that R be bounded by a Jordan configuration B, that $u(x, y)$ be harmonic and bounded in R, continuous in $R + B$ except perhaps for a finite number of points, $u(x, y)$ not identically equal to $u(x, -y)$ in R, and